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Nonisotropic Solutions of the Boltzmann Equation

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We consider the relaxation to equilibrium of a spatially uniform Maxwellian gas. We expand the solution of the nonlinear Boltzmann equation in a truncated series of orthogonal functions. We integrate numerically the equation for nonisotropic initial conditions. For certain simple conditions we find interesting proximity effects and other transient relaxation phenomena at thermal energies. Furthermore, we define a resummation of the orthogonal expansion which is more convenient than the original one for the numerical analysis of the relaxation process.

KEY WORDS: Boltzmann equation; nonisotropic initial conditions; Maxwell molecules; moment equations; numerical calculations.

1. INTRODUCTION

We consider a spatially uniform gas of structureless particles which interact through binary elastic collisions. We look for the corresponding one-particle distribution function $f(p, t)$. Its temporal evolution is characterized by the nonlinear Boltzmann equation. Research on this equation was induced by the lack of an explicit solution for the associated initial and boundary value problems. A great variety of interaction models considered in the literature give insight into this Cauchy problem.[(1,2)] Maxwell models provide a large simplification of the Boltzmann equation. For these models the collision probability depends on the scattering angle and not on the relative momentum. In this case the general solution is known for isotropic initial conditions within a certain Hilbert space. It is given as an orthogonal polynomial expansion, with time-dependent coefficients which

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obey a solvable coupled set of nonlinear equations.[2] This isotropic solution has been extensively studied by analytical resolution of the coupled system up to some order and the numerical evaluation of the resulting truncated series. Many relaxation phenomena have been found from this numerical analysis. The Tjon overshoot phenomenon, the oscillatory relaxation to equilibrium, and other proximity effects have demonstrated that the orthogonal expansion is very suitable for the numerical analysis of the Boltzmann equation.[2]

Research on the exact time-dependent solutions of the Boltzmann equation has been mainly limited to isotropic velocity distributions. The analysis of the anisotropic case has been put aside in view of its complex mathematical structure. Besides a few pioneering works,[3,4] little has been done toward solving this problem. For instance, a numerical study is lacking. Recently the orthogonal expansion of the distribution function has been generalized to include nonisotropic initial conditions.[5] This has been done by using spherical coordinates, and finding a recursively soluble set of equations for the corresponding generalized moments. Previous methods[6,7] which use tensorial moments with multiindices are rather elegant, but inadequate for a numerical analysis.

In this paper we develope a analysis of the nonlinear Boltzmann equation in a two-dimensional velocity space, for nonisotropic initial conditions. The solution is represented by a truncated expansion in orthogonal functions (Section 2). This procedure is restricted to moderate values of the energy, where not too many terms in the series are required. However, we are able to show different relaxation features for the anisotropic case in Section 3. We generalize the criterion of Hauge[8] and Alexanian,[9] formulating conditions on the initial distribution function which determine the basic features of the final approach to equilibrium. In Section 4 we resum the expansion in orthogonal functions and obtain a different representation of the distribution function, which can be more convenient than the original one for numerical studies of the relaxation process. We conclude with a discussion of the results.

2. GENERAL SOLUTION OF BOLTZMANN EQUATION

The spatially uniform nonlinear Boltzmann equation in d dimension is[2]

$$\frac{\partial}{\partial t} f(\mathbf{p}, t) = \int \frac{g}{m} \sigma(g, \hat{g} \cdot \hat{n}) [f(\mathbf{p}', t) f(\mathbf{p}'_1, t) - f(\mathbf{p}, t) f(\mathbf{p}_1, t)] \, d\hat{n} \, d\mathbf{p}_1 \tag{2.1}$$

The incoming and postcollisional momenta are related by the dynamics

$$\mathbf{p}' = \tfrac{1}{2}(\mathbf{p} + \mathbf{p}_1) - \tfrac{1}{2}|\mathbf{p} - \mathbf{p}_1|\,\hat{n} \tag{2.2a}$$

$$\mathbf{p}_1' = \tfrac{1}{2}(\mathbf{p} + \mathbf{p}_1) + \tfrac{1}{2}|\mathbf{p} - \mathbf{p}_1|\,\hat{n} \tag{2.2b}$$

$\mathbf{g} = \mathbf{p}_1 - \mathbf{p}$ and $\mathbf{g}' = \mathbf{p}_1' - \mathbf{p}'$ are the relative momenta of the particles before and after the collision with cross section $\sigma(g, \hat{g}\cdot\hat{n})$, and $\hat{n}$ is a unit vector in the direction of $\mathbf{g}'$.

The H theorem guarantees that $f(\mathbf{p}, t)$ approaches its equilibrium value for large times:

$$f(\mathbf{p}, t) \xrightarrow[t \to \infty]{} f_0(p) = \frac{\eta}{(2\pi mkT)^{d/2}} \exp\left(-\frac{p^2}{2mkT}\right) \tag{2.3}$$

where η is the number density and k is Boltzmann's constant. The temperature T defines the average energy per degree of freedom. The temporal evolution of the gas is observed from the center-of-mass reference frame.

The purpose of this work is to tackle the anisotropic Boltzmann equation for an interaction model with a momentum-independent collision probability (Maxwell model):

$$g\sigma(g, \hat{g}\cdot\hat{n}) = \alpha(\hat{g}\cdot\hat{n}) \tag{2.4}$$

Here we will analyze the two-dimensional case $d = 2$. The more complicated three-dimensional case has similar characteristics.[5]

The distribution function can be expanded in a nonisotropic Laguerre series,[5] when its moments exist:

$$f(\mathbf{p}, t) = f_0(p)[1 + R(\varepsilon, \theta, t)] \tag{2.5a}$$

$$R(\varepsilon, \theta, t) = \sum_{n=1}^{\infty} \sum_{q=0}^{n} C_{nq}(t)\, R_{nq}(\varepsilon, \theta) \tag{2.5b}$$

with θ the polar angle of the momentum $\mathbf{p}$, and $\varepsilon = p^2/2mkT$ the energy per thermal unit. The functions

$$\begin{aligned} R_{nq}(\varepsilon, \theta) = {} & (-1)^{(n + |2q-n|)/2} \left(\frac{n - |2q-n|}{2}\right)! \\ & \times \varepsilon^{(|2q-n|)/2}\, L^{|2q-n|}_{(n-|2q-n|)/2}(\varepsilon)\, e^{i(2q-n)\theta} \end{aligned} \tag{2.6}$$

form a complete set in a Hilbert space $\mathscr{L}_2$ with norm

$$\|R\|^2 = \int e^{-\varepsilon} |R(\varepsilon, \theta)|^2 \, d\varepsilon\, d\theta \tag{2.7}$$

They are orthogonal in the following sense:

$$\int e^{-\varepsilon} R^*_{nq}(\varepsilon, \theta)\, R_{n'q'}(\varepsilon, \theta)\, d\varepsilon\, d\theta$$

$$= 2\pi \left(\frac{n + |2q-n|}{2}\right)! \left(\frac{n - |2q-n|}{2}\right)! \,\delta_{nn'}\,\delta_{qq'} \tag{2.8}$$

The coefficients C_{nq} are generalized moments of the distribution function,

$$C_{nq}(t) = \frac{1/\eta}{[(n+|2q-n|)/2]!\,[(n-|2q-n|)/2]!} \times \int f(\mathbf{p}, t)\, R_{nq}\left(\frac{p^2}{2mkT}, \theta\right) d^2\mathbf{p} \tag{2.9}$$

Conservation of particles, momentum, and energy requires

$$C_{00}(t) = 1 \tag{2.10a}$$

$$C_{10}(t) = C_{11}(t) = C_{21}(t) = 0 \tag{2.10b}$$

Since $R(\varepsilon, \theta, t)$ is a real-valued function,

$$C_{nq}(t) = C^*_{n,n-q}(t) \tag{2.11}$$

Substituting expansion (2.5) into Eq. (2.1), we obtain a recursive set of equations for the moments $C_{nq}(t)$,[5]

$$C_{00}(t) = 1 \tag{2.12a}$$

$$C_{1Q}(t) = C_{21}(t) = 0 \tag{2.12b}$$

$$C_{2Q}(t) = C_{2Q}(0)\, e^{-\Lambda_{2Q} t} \tag{2.12c}$$

$$C_{3Q}(t) = C_{3Q}(0)\, e^{-\Lambda_{3Q} t} \tag{2.12d}$$

$$C_{NQ}(t) = C_{NQ}(0)\, e^{-\Lambda_{NQ} t} + \int_0^t e^{-\Lambda_{NQ}(t-\tau)} \sum_{n=2}^{N-2} \sum_{q=q_0}^{q_1} (-1)^q \mu_n^{NQ} \times C_{nq}(\tau)\, C_{N-n,Q-q}(\tau)\, d\tau \tag{2.12e}$$

with

$$q_0 = \max(0, Q-N+n) \tag{2.13a}$$

$$q_1 = \min(n, Q) \tag{2.13b}$$

The real coefficients μ_n^{NQ} are given by

$$\mu_n^{NQ} = \eta \int_0^{2\mu} \alpha(\text{cps } \theta) \left[\cos\frac{\theta}{2}\right]^{N-n} \left[\sin\frac{\theta}{2}\right]^n \times \exp\left\{i\left(Q - \frac{N}{2}\right)\theta + n\frac{\pi}{2}\right]\right\} d\theta \tag{2.14}$$

and Λ_{NQ} are the eigenvalues of the linearized Boltzmann equation,

$$\begin{aligned} \Lambda_{NQ} &= \mu_0^{00}(1 + \delta_{N0}) - \mu_0^{NQ} - (-1)^Q \mu_N^{NQ} \\ &= \eta \int_0^{2\pi} \alpha(\cos\theta) \left[1 + \delta_{N0} \right.\\ &\quad \left. - e^{i(Q-N/2)\theta}\left(\cos^N\frac{\theta}{2} + e^{i(Q+N/2)\theta}\sin^N\frac{\theta}{2}\right)\right] d\theta \end{aligned} \tag{2.15}$$

Equations (2.5) and (2.12) provide the general solution for the two-dimensional Maxwell models. Several authors have investigated these solutions,[5,10,11] finding sufficient conditions for the existence and absolute convergence of the expansion (2.6).

For isotropic initial conditions Eq. (2.6) reduces to the known Laguerre series[2]

$$f(\mathbf{p}, t) = f_0(p) \sum_{n=0}^{\infty} c_n(t)\, L_n\left(\frac{p^2}{2mkT}\right) \tag{2.16}$$

where the moments c_n satisfy

$$c_0(t) = 1 \tag{2.17a}$$

$$c_1(t) = 0 \tag{2.17b}$$

$$c_n(t) = e^{-\lambda_n t}\left[c_n(0) + \int_0^t e^{\lambda_n \tau} \sum_{m=1}^{n-1} \mu_{nm} c_m(\tau)\, c_{n-m}(\tau)\right] d\tau \tag{2.17c}$$

with

$$c_n(t) = \frac{(-1)^n}{n!} C_{2n,n}(t) \tag{2.18}$$

$$\lambda_n = \Lambda_{2n,n} \tag{2.19}$$

$$\mu_{nm} = (-1)^m \binom{n}{m} \mu_m^{2n,n} \tag{2.20}$$

Comparison of Eq. (2.14) with Eq. (2.20) shows that the anisotropic coefficients μ_n^{NQ} can be obtained from the corresponding isotropic coefficients μ_{nm},

$$\mu_n^{NQ} = [\mathrm{sgn}(2Q-N)\}^n \sum_{m=m_0}^{m_1} \binom{|2Q-N|}{2m-n} \binom{(N+|2Q-N|)/2}{m}^{-1} (-1)^m$$
$$\times \mu_{(N+|2Q-N|)/2,m} \tag{2.21a}$$

$$m_0 = \left[\frac{n+1}{2}\right] \tag{2.21b}$$

$$m_1 = \left[\frac{|2Q-N|+n}{2}\right] \tag{2.21c}$$

where $[X]$ is the largest integer less than or equal to X and $\mathrm{sgn}(X)$ is -1 when X is negative or $+1$ when X is positive or zero. The particular features of a Maxwell model are resummed in the corresponding set of coefficients μ_{nm}.

From Eq. (2.14) the following symmetry for the anisotropic coefficients is obtained:

$$\mu_n^{NQ} = (-1)^n \mu_n^{N,N-Q} \tag{2.22}$$

Therefore

$$\Lambda_{NQ} = \Lambda_{N,N-Q} \tag{2.23}$$

When $\alpha(x) = \alpha(-x)$ we have

$$\mu_n^{NQ} = (-1)^Q \mu_{N-n}^{NQ} \tag{2.24}$$

The eigenvalues Λ_{NQ} can be written down in terms of Chebyshev polynomials of the first kind,

$$\Lambda_{NQ} = \eta \int_0^{2\pi} d\theta\, \alpha(\cos\theta) \left[1 + \delta_{N0} \right.$$
$$\left. - \left(\cos\frac{\theta}{2}\right)^N T_{|2Q-N|}\left(\cos\frac{\theta}{2}\right) - \left(\sin\frac{\theta}{2}\right)^N T_{|2Q-N|}\left(\sin\frac{\theta}{2}\right) \right] \tag{2.25}$$

By substituting the explicit expression of $T_n(x)$,[12]

$$T_n(x) = \sum_{m=0}^{[n/2]} a_{nm} x^{n-2m} \tag{2.26a}$$

$$a_{nm} = (-1)^m \frac{n\Gamma(n-m)}{m!\,(n-2m)!} 2^{n-2m-1} \tag{2.26b}$$

it is possible to relate Λ_{NQ} to the corresponding isotropic eigenvalues λ_n:

$$\Lambda_{NQ} = \sum_{m=0}^{[|2Q-N|/2]} a_{|2Q-N|,m}\, \lambda_{(|2Q-N|+N)/2-m} \tag{2.27}$$

Then

$$\Lambda_{N,[N/2]} = \Lambda_{N,[(N+1)/2]} = \lambda_{[(N+1)/2]} \tag{2.28}$$

Furthermore, the recurrence relation for the Chebyshev polynomials leads to

$$2\Lambda_{N+1,Q} = \Lambda_{NQ} + \Lambda_{N,Q-1}; \qquad Q \neq 0, N+1 \tag{2.29}$$

This recurrence equation shows that the relation $3\lambda_2 = 2\lambda_3$ of the isotropic case leads to the folowing multiple degeneracy:

$$3\Lambda_{31} = 3\Lambda_{32} = 3\Lambda_{42} = 2\Lambda_{52} = 2\Lambda_{53} = 2\Lambda_{63} \tag{2.30}$$

3. NUMERICAL CALCULATIONS

Expressions (2.5) and (2.12) give the general solution of the non-isotropic two-dimensional Boltzmann equation for Maxwell models, provided the series converges. By using the fact that the moments $C_{NQ}(t)$ are a sum of exponential transients, namely $C_{NQ}(t) = \sum_n a_n e^{-b_n t}$, we can formally define

$$C_{NQ}(t) \equiv \begin{pmatrix} b_1 & b_2 \cdots b_s \\ a_1 & b_2 \cdots a_s \end{pmatrix} \tag{3.1}$$

with $b_n < b_{n+1}$. Therefore, operations between moments reduce to simple algebraic matrix calculations. For instance, integration is given by

$$\int_0^t C_{NQ}(\tau)\, d\tau \equiv \begin{pmatrix} 0 & b_1 & \cdots & b_s \\ \sum_{j=1}^{s} \frac{a_j}{b_j} & -\frac{a_1}{b_1} & \cdots & -\frac{a_s}{b_s} \end{pmatrix} \tag{3.2}$$

This idea enables us to develop an algorithm for calculating each moment C_{NQ} up to $N \gtrsim 30$.

We test the precision of our numerical method through the BKW mode,[10,13]

$$f(\mathbf{p}, t) = f(p)\frac{e^{-\sigma\varepsilon/(1-\sigma)}}{1-\sigma}\left[1 - \frac{\sigma}{1-\sigma}\left(1 - \frac{\varepsilon}{1-\sigma}\right)\right]; \qquad \varepsilon = \frac{p^2}{2mkT} \tag{3.3a}$$

$$\sigma(t) = \sigma(0)\, e^{-\Lambda_{42}t/2}; \qquad 0 \leqslant \sigma(0) \leqslant 1/2 \tag{3.3b}$$

which is the only known exact nontrivial solution. The size of the relative error in the evaluation of the moments $C_{NQ}(t)$ is smaller than 10^{-12}. Furthermore, we have evaluated the relative error for various values of energy and time. Figure 1 shows the relative error of our numerical results for a BKW solution with $\sigma(0)=0.25$ for the Tjon–Wu interaction model,[2]

$$\alpha(\hat{g}\cdot\hat{n})=(\mu/4\eta)[1-(\hat{g}\cdot\hat{n})^2]^{1/2} \tag{3.4}$$

The truncation order N was arbitrarily fixed at $N_0=12$. We see that even at $t=0$ the errors are less than 1% for energies below $\varepsilon=20$. The energy range where it is possible to attain a good approximation is one or two orders above the thermal range $\varepsilon\approx 1$, and it increases almost linearly with time.

As a first example, we will consider a class of initial distributions with few nonvanishing moments, as introduced by Barnsley and Cornille[14] for the isotropic case. We shall generalize this class of solutions for the anisotropic case. The greatest difficulty with an initially truncated moment expansion

$$R(\varepsilon,\theta,0)=\sum_{n=1}^{N}\sum_{q=0}^{n} C_{nq}(0)\,R_{nq}(\varepsilon,\theta) \tag{3.5}$$

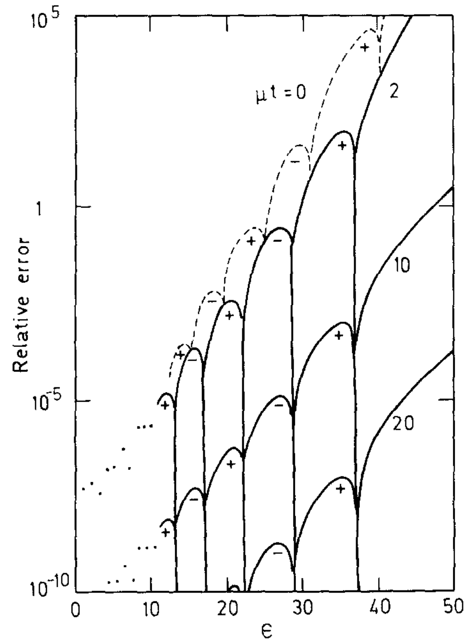


Fig. 1. Relative error for the numerical evaluation of the BKW mode with $\sigma(0)=0.25$ and the Tjon–Wu interaction model. The truncation order is $N_0=12$.

is to guarantee its positivity $1+R(\varepsilon, \theta, 0) \geqslant 0$. For large values of energy $R_{nq} \propto \varepsilon^n$ and

$$R(\varepsilon, \theta, 0) \approx \varepsilon^N \sum_{q=0}^{n} C_{nq}(0) e^{i(2q-n)\theta}; \qquad \varepsilon \gg 1 \tag{3.6}$$

Then the positivity condition requires, besides an even truncation order, the presence of an angle-independent term with moment

$$C_{N,N/2}(0) \geqslant \frac{1}{2} \sum_{q=0}^{N} |C_{Nq}(0)| \tag{3.7}$$

This condition ensures that the high-energy tail is initially positive and overpopulated; but it does not guarantee the positivity at lower energies. However, when all moments are null, $R(\varepsilon, \theta, 0)=0$, and we can always choose $C_{Nq}(0)$ sufficiently small in order to satisfy the positivity condition. The simplest anisotropic initial condition is given by

$$\begin{aligned} R(\varepsilon, \theta, 0) = {} & C_{N,N/2}(0)\, R_{N,N/2}(\varepsilon, \theta) \\ & + C_{nq}(0)[R_{nq}(\varepsilon, \theta) + R^*_{nq}(\varepsilon, \theta)] \end{aligned} \tag{3.8}$$

We have arbitrarily chosen $C_{nq}(0)$, with any $n \leqslant N$ and $q \neq N/2$, as a real-valued and positive constant, through a change of the angular phase. Now condition (3.7) reads

$$C_{N,N/2}(0) \geqslant 2\delta_{Nn} C_{nq}(0) \tag{3.9}$$

When $C_{nq}(0)=0$ this "simple anisotropic" (SA) solution reduces to the isotropic fundamental positive solution of Barnsley and Cornille. Condition (3.9) ensures that the high-energy tail is initially overpopulated. Furthermore, this SA initial condition has a $|2q-n|$-order rotational symmetry. The evolution of the distribution function has to conserve this symmetry. Then moments $C_{n'q'}$ with $|2q'-n'|$ multiple of $|2q-n|$ (or zero) can be populated. These, initially real, moments must remain real at all times. This condition is intimately related to the conservation of a reflection symmetry.

For $N=2$ there is no such simple initial condition in view that $C_{21}(t)=0$. For $N=4$ there are some SA solutions, for instance,

$$R(\varepsilon, \theta, 0) = 2C_{42}(0)\, L_2^0(\varepsilon) - 2C_{41}(0)\, \varepsilon L_1^2(\varepsilon) \cos(2\theta) \tag{3.10a}$$

$$C_{42}(0)=0.16; \qquad C_{41}(0)=0.04 \tag{3.10b}$$

This initial SA distribution function displays an underpopulated ring at $1 \lesssim \varepsilon \lesssim 4$, as shown in Fig. 2. The evolution of this energy region was

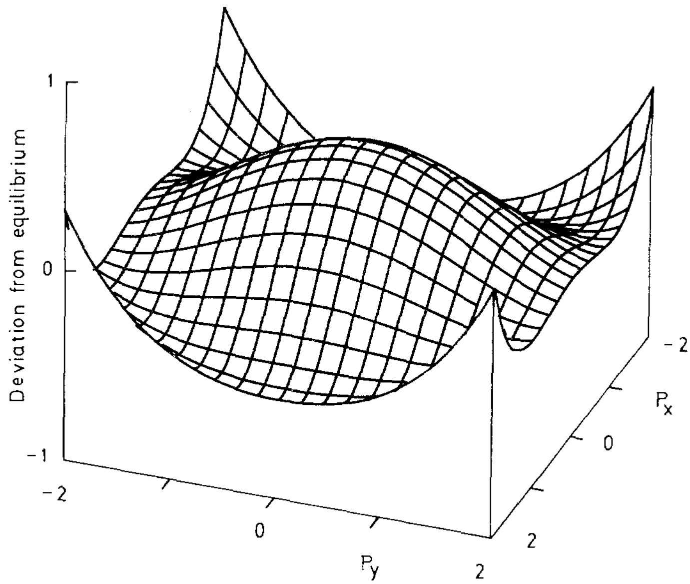


Fig. 2. Initial deviation from the equilibrium $R(\varepsilon, \theta, 0)$ for the simple anisotropic solution (3.10). The momentum is indicated in reduced units $p/(mkT)^{1/2}$.

calculated via Eq. (2.5) truncated at $N_0 = 32$ for various times and two different angles. Numerical convergence was found to be particularly good in the sense that small variations of the truncation order N_0 do not alter significantly the previous results. At $\theta = 0°$ the distribution function exhibits a monotonic approach toward equilibrium. However, an interesting relaxation phenomenon occurs at $\theta = 90°$ (Fig. 3): As time elapses, the high-energy tail turns over the underpopulated region, leading to a transient overpopulation effect at thermal energies. This "proximity" effect was first observed by Barnsley and Cornille for an isotropic fundamental positive solution.[14]

In view of their simplicity, the SA solutions are quite adequate to analyze the characteristic features of the relaxation to equilibrium. The distribution function may relax toward equilibrium monotonically, or display a nonmonotonic behavior. Furthermore, these monotonic relaxation and transient overpopulation and depopulation effects may occur simultaneously for different angles for a given initial distribution function and even for the same energy. This is shown in Fig. 4 for the following combination of two SA solutions:

$$R(\varepsilon, \theta, 0) = -\frac{2}{73}\left[6L_3^0(\varepsilon) - \sqrt{\varepsilon}\, L_1^{\prime}(\varepsilon)\cos\theta - \varepsilon L_1^2(\varepsilon)\cos 2\theta\right] \quad (3.11)$$

For $\varepsilon = 6.2$ this distribution function exhibits simultaneous transient under-

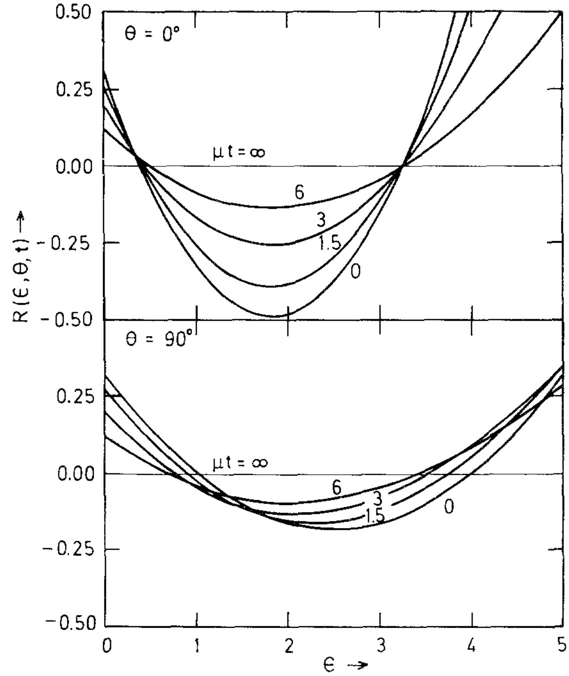


Fig. 3. Deviation from equilibrium $R(\varepsilon, \theta, t)$ for the initial condition of Fig. 2 with the Tjon–Wu interaction model.

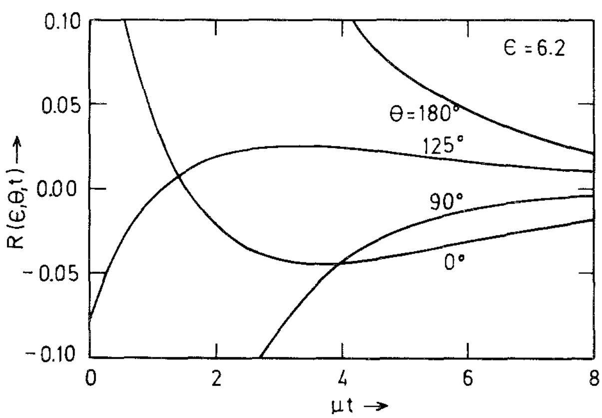


Fig. 4. Time evolution of $R(\varepsilon, \theta, t)$ for $\varepsilon = 6.2$ and different values of the angle in the Tjon–Wu interaction model. The initial condition is given by Eq. (3.11).

population and overpopulation effects at $\theta = 0°$ and $125°$, respectively. Furthermore, it displays a monotonic relaxation from above at $\theta = 180°$ and from below at $\theta = 90°$. These phenomena show quite clearly the increasingly complex features of the anisotropic relaxation process in comparison with the relatively simple isotropic situation.[15]

It is worthwhile to analyze the dependence of the final approach to equilibrium upon the preparation of the initial distribution function. Expansion (2.5) provides a useful hint: At large times the dominant contribution to expansion (2.5) is given by the slowest decaying moments, namely

$$R(\varepsilon, \theta, t) \approx e^{-\Lambda_{31} t}\{2C_{42}(0)\, L_2^0(\varepsilon) + [C_{31}(0) e^{i\theta} + C_{31}^*(0) e^{-i\theta}] \sqrt{\varepsilon}\, L_1^1(\varepsilon)\} \tag{3.12}$$

Then the relaxation to equilibrium is determined by the sign of the function in the curly brackets. For a large fixed value of the energy,

$$R(\varepsilon, \theta, t) \approx e^{-\Lambda_{31} t}\, C_{42}(0)\, \varepsilon^2 \tag{3.13}$$

and the well-known criterion of Hauge[8] and Alexanian[9] is obtained: At large values of the energy, the relaxation to equilibrium will be from above or below, depending on whether $C_{42}(0)$ is positive or negative, respectively. This criterion also can be applied to the anisotropic case. Furthermore, the final approach to equilibrium of the high-energy tail is independent of the angle variable when $C_{42}(0)$ is nonvanishing. Actually, the SA distribution function of Fig. 2 is characterized by a positive $C_{42}(0)$ moment and a monotonic overpopulated approach to equilibrium is obtained at large energies. In general the positivity condition (3.7) for a SA solution with a nonzero $C_{42}(0)$ moment leads to an overpopulation approach to equilibrium of the high-energy tail. When $C_{42}(0)$ is null, the criterion applies to the slowest decaying nonzero moment. For instance, the final approach to equilibrium of an initial condition given by $C_{31}(0) = C_{32}(0) = C_{63}(0) = 1/30$ is controlled by the moment $C_{31}(0)$. This is consistent with the purely anisotropic relaxation behavior displayed in Fig. 5: At $\theta = 0°$ the third zero of $R(\varepsilon, \theta, t)$ evolves toward the right, leading to a transient underpopulation effect. On the other hand, the high-energy tail displays a monotonic overpopulated approach to equilibrium at $\theta = 180°$. At $\theta = 90°$ the $n = 3$ terms are null and the $n = 6$, $q = 3$ term dominates the monotonic time evolution.

4. MODIFIED LAGUERRE EXPANSION

The simple anisotropic solutions studied in Section 3 show that the momentum expansion (2.5) has a good convergence for initial distribution

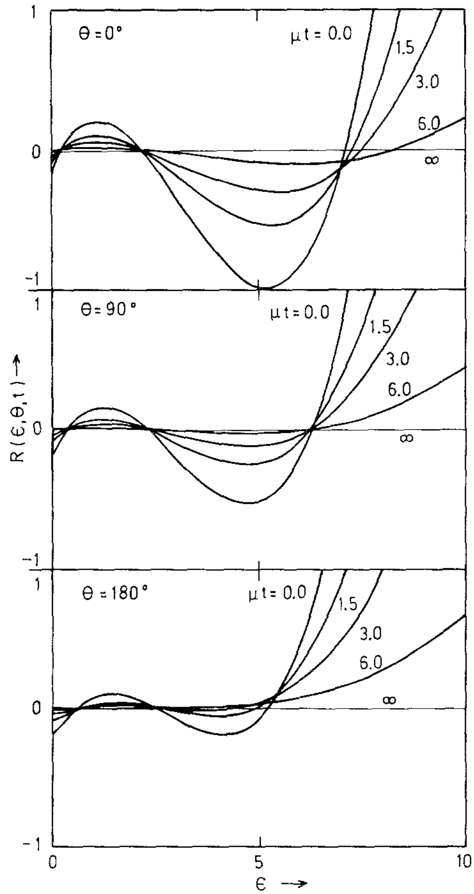


Fig. 5. Deviation from equilibrium in the Tjon–Wu model for an initial condition (3.5) with $C_{31} = C_{32} = C_{63} = 1/30$.

functions overpopulated at high energies. We have shown that in that case, as the BKW mode, the expansion (2.5) converges in a large energy range. However, the situation is particularly difficult when we analyze the relaxation features of a distribution function with a scarce high-energy tail. This follows from Eq. (3.6), which shows that the truncated expansion gives an approximation overpopulate at high energies. Therefore, it is necessary to improve the convergence, particularly for low values of time and high energies. Padé approximations are worthwhile techniques to improve convergence in the isotropic case.[15,16]

Another method is suggested by the BKW mode: A better description of a distribution function with an underpopulated high-energy tail will be achieved when expansion (2.5) is multiplied by an adequate exponential weighting factor $\exp[-\sigma\varepsilon/(1-\sigma)]$. Such a factor would be useful at $t=0$, but it may become a catastrophic hindrance for the series convergence at longer times. This difficulty may be solved by means of a time-dependent parameter $\sigma(t)$. An appropriate choice of the asymptotic behavior of $\sigma(t)$ will enable a good approximate description of the relaxation toward equilibrium. This modified moment expansion was employed by several authors for the isotropic case.[2,9,17] It has been also found from an iterative solution of the BE for the VHP model.[18] Our purpose here is to introduce a similar anisotropic modified expansion.

This goal could be achieved by the application of Bobylev symmetry[10] to the Fourier-transformed distribution function, as in the isotropic case.[2] We shall use a straightforward approach. We rewrite the Laguerre expansion (2.5) as in the BKW mode, Eq. (3.3):

$$f(\mathbf{p}, t)=f_0(p)\frac{e^{-\sigma\varepsilon/(1-\sigma)}}{1-\sigma(t)}\sum_{n=0}^{\infty}\sum_{q=0}^{n} C_{nq}(0)[1-\sigma] \times e^{\sigma\varepsilon/(1-\sigma)}\, R_{nq}(\varepsilon, \theta) \tag{4.1}$$

with $\sigma=\sigma(t)$ an arbitrary function of time. The functions $R_{nq}(\varepsilon, \theta)$ satisfy the following property:

$$(1-\sigma)\, e^{\sigma\varepsilon/(1-\sigma)}\, R_{nq}(\varepsilon, \theta) = \frac{1}{(1-\sigma)^{n/2}}\sum_{m=0}^{\infty}\frac{\sigma^m}{(1-\sigma)^m}\frac{1}{m!}R_{n+2m,q+m}\left(\frac{\varepsilon}{1-\sigma}, \theta\right) \tag{4.2}$$

Replacing in Eq. (4.1)

$$f(\mathbf{p}, t)=f_0(p)\frac{e^{-\sigma\varepsilon/(1-\sigma)}}{1-\sigma}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{\gamma_{nq}(t)}{(1-\sigma)^{n/2}}R_{nq}\left(\frac{\varepsilon}{1-\sigma}, \theta\right) \tag{4.3}$$

where

$$\gamma_{nq}(t)=\sum_{m=0}^{(n-|2q-n|)/2}\frac{\sigma^m}{m!}C_{n-2m,q-m}(t) \tag{4.4}$$

Conservation of particles, momentum, and energy requires

$$\gamma_{00}(t)=1 \tag{4.5a}$$

$$\gamma_{1Q}(t)=0 \tag{4.5b}$$

$$\gamma_{21}(t)=\sigma(t) \tag{4.5c}$$

Actually, Eq. (4.3) represents a modified Laguerre expansion of the distribution function. It reads

$$f(\mathbf{p}, t) = f_{\gamma_{21}(t)}(p)\left[1 + \sum_{n=1}^{\infty} \sum_{q=0}^{n} \frac{\gamma_{21}(t)}{[1-\gamma_{21}(t)]^{n/2}} \times R_{nq}\left(\frac{\varepsilon}{1-\gamma_{21}(t)}, \theta\right)\right] \tag{4.6}$$

with

$$f_\sigma(p) = \frac{\eta}{2\pi mkT(1-\sigma)} \exp\left(-\frac{p^2}{2mkT(1-\sigma)}\right) \tag{4.7}$$

which is the equilibrium distribution function (2.4) with a modified temperature $(1-\sigma)T$.

The coefficients γ_{nq} are generalized moments of $f(\mathbf{p}, t)$:

$$\gamma_{nq}(t) = \frac{[1-\gamma_{21}(t)]^{n/2}}{[(n+|2q-n|)/2]!\,[(n-|2q-n|)/2]!}\frac{1}{\eta} \times \int f(\mathbf{p}, t)\, R_{nq}\left(\frac{p^2/2mkT}{1-\gamma_{21}(t)}, \theta\right) d^2\mathbf{p} \tag{4.8}$$

with the following properties:

$$\gamma_{nq}(t) = \gamma^*_{n,n-q}(t) \tag{4.9a}$$

$$\gamma_{nq}(t) \xrightarrow[t\to\infty]{} \delta_{n,2q}\frac{\gamma_{21}(\infty)^q}{q!} \tag{4.9b}$$

Inserting expansion (4.6) in Eq. (2.1), we obtain the following infinite set of equations for the moments:

$$\gamma_{00}(t) = 1 \tag{4.10a}$$

$$\gamma_{1Q}(t) = 0 \tag{4.10b}$$

$$\gamma_{2Q}(t) = \gamma_{2Q}(0)e^{-\Lambda_{2Q}t} \quad \gamma_{21}(t) \text{ arbitrary} \tag{4.10c}$$

$$\gamma_{3Q}(t) = \gamma_{3Q}(0)\, e^{-\Lambda_{3Q}t} \tag{4.10d}$$

$$\gamma_{NQ}(t) = \gamma_{NQ}(0)\, e^{-\Lambda_{NQ}t} + \int_0^t e^{-\Lambda_{NQ}(t-\tau)}\left[\gamma_{21}(\tau)\,\gamma_{N-2,Q-1}(\tau) + \sum_{n=2}^{N-2} \sum_{q\leqslant q_0}^{q_1} (-1)^q \mu_n^{NQ} \gamma_{nq}(\tau)\,\gamma_{N-n,Q-q}(\tau)\right] d\tau \tag{4.10e}$$

This expression is an alternative representation of the general solution of the two-dimensional nonlinear Boltzmann equation for Maxwell models. The choice $\gamma_{21} = 0$ leads to the usual moment expansion (2.5). It is worthwhile to note that a criterion for a good choice of $\gamma_{21}(t) < 1$ has to be formulated. For instance, the choice $\gamma_{21}(t) = \sigma \exp(-\Lambda_{42} t/2)$ for the initial condition $\gamma_{NQ}(t) = 0$ for other N and Q is a simple exact solution, namely the BKW mode.

Henceforth, we shall analyze the temporal evolution of an initially truncated modified Laguerre serie with only two nonvanishing moments. Such an initial condition is very suitable to applying the previous concepts. We choose $2\gamma_{21} = -4\gamma_{20} = -\gamma_{22} = 1$, namely

$$R(\varepsilon, \theta, 0) = 4\varepsilon e^{-\varepsilon}[1 - \cos(2\theta)] - 1 \tag{4.11}$$

This initial distribution displays two overpopulated peaks at $\varepsilon = 1$ with $\theta = 90°$ and $270°$, respectively. It is underpopulated at high energies, in contrast to the SA distributions of Section 3. Furthermore, as $C_{42}(0) = -1/8$, the criterion of Hauge and Alexanian guarantees that the high-energy tail will display a monotonic relaxation to equilibrium from below. The temporal evolution of this distribution was numerically evaluated with the usual Laguerre expansion (2.5) truncated at $N_0 = 16$. In Fig. 6 we show the absolute error of our numerical evaluation for the initial condition (4.11). This error is small in the region of interest $\varepsilon \lesssim 4$. Actually, the energy range where the approximation is good increases with time. This ensures our conclusions.

Figure 7 shows the distribution (4.11) at four different times, namely $\mu t = 0$, 1, 2, and 4. It displays quite clearly a new and interesting relaxation

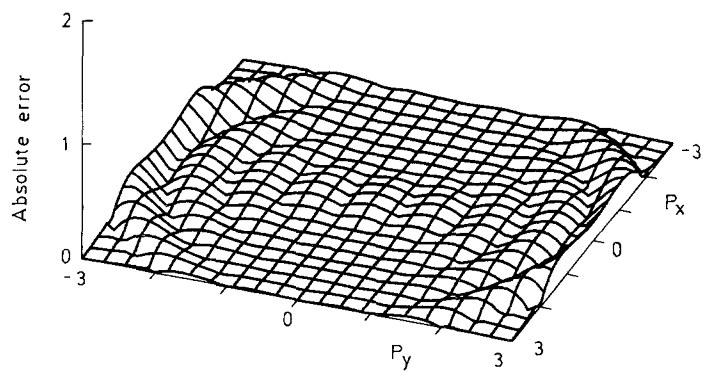
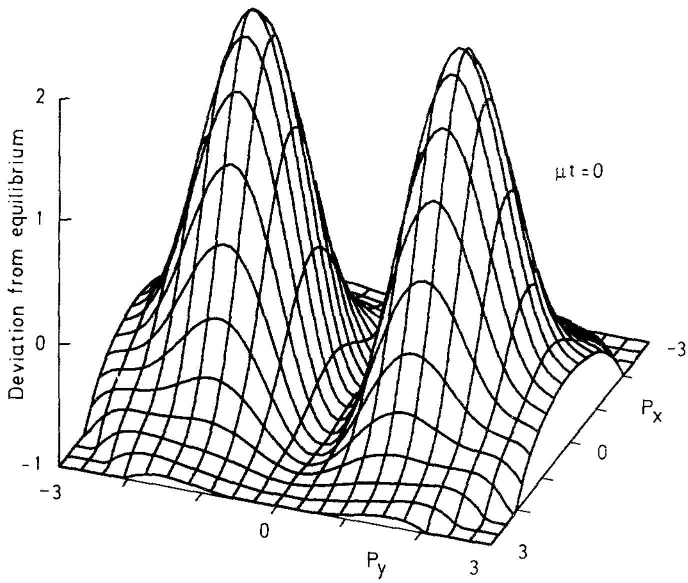
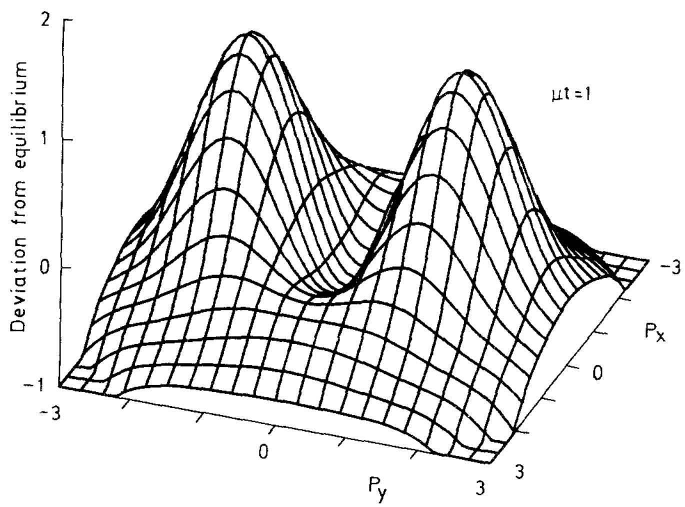


Fig. 6. Absolute error in the numerical evaluation of the initial condition (4.11). The truncation order is $N_0 = 16$.

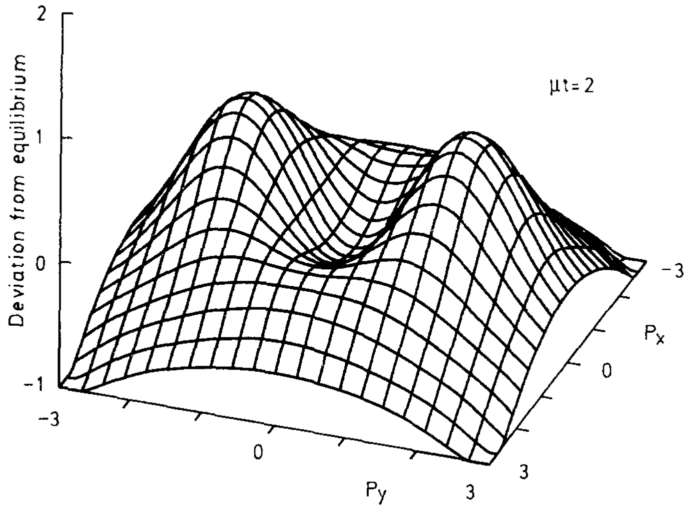


(a)

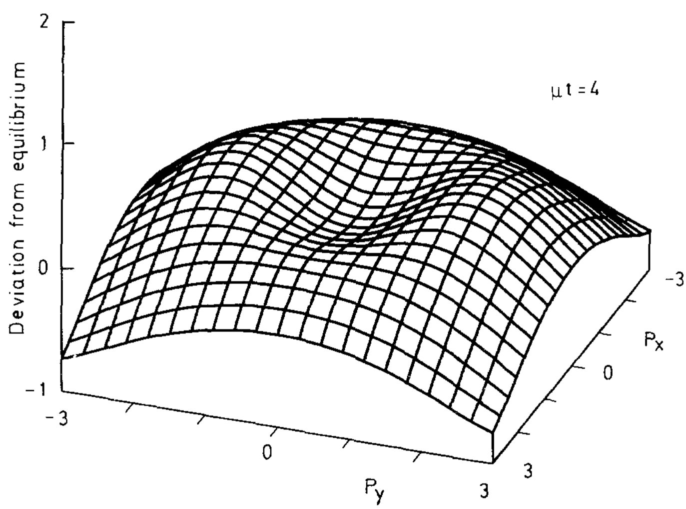


(b)

Fig. 7. Time evolution of the deviation from equilibrium for the initial condition (4.11). The momentum is indicated in reduced units $p/(mkT)^{1/2}$.



(c)



(d)

Fig. 7 (*continued*)

phenomenon: The overpopulated peaks show a preferential spread in angular direction, giving rise to a population ring at $\varepsilon = 1$. This "preferential spreading effect" may be interpreted as follows: the majority of the particles are initially restricted to a very small region in momentum space with $\varepsilon \approx 1$. As time elapses, the energy and momentum conservation laws (2.3) tend to foce these particles to relax very slowly in energy, but with a fast angular spreading.

5. CONCLUSIONS

In the present paper we have studied the two-dimensional in velocity nonisotropic Boltzmann equation for Maxwell interaction models. We expanded its solution in a truncated series of orthogonal functions with time-dependent coefficients given by an analytically solvable set of equations. This orthogonal expansion is very suitable for the numerical analysis of the relaxation process, when restricted to moderate values of the energy, where not too many terms are required. We showed interesting proximity effects and other transient relaxation phenomena. For instance, many transient overpopulation and depopulation effects may occur simultaneously for different angles at the same energy. The basic features of the final approach to equilibrium are determined by the criterion of Hauge and Alexanian for the isotropic case. However, a purely anisotropic relaxation proces may occur when the moment $C_{42}(0)$ is null. Finally, we define a reummation of the orthogonal expanion which can be more convenient than the original one for the numerical analyis of the relaxation process. Even though most of our discusion is restricted to two-dimensional initial conditions, which represent the simplest anisotropic model, it is worthwhile to note that our numerical method can be applied to case of arbitrary dimension.

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